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A simple relativistic theory of gravitation

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Abstract. It is shown that a particular theory with a Lorentz-invariant scalar potential ϕ and a simple gravitational metric involving ϕ leads to the same results for the three Einstein tests as General Relativity.

1. Introduction

In recent years there has been a revival of interest in relativistic theories of gravitation other than General Relativity, notably those of the Lorentz-invariant type. Efforts have been directed towards comparing results obtained to order $1/c^2$ from these theories with the three classical tests of General Relativity. In a detailed comparative analysis by Whitrow and Murdoch (1965) it was shown that a generalization of Nordström's (1912) scalar theory that included as special cases theories due to Littlewood (1953), Bergmann (1956) and others gave the same result as General Relativity for the gravitational red-shift, and for a particular value of an arbitrary parameter the same formula as General Relativity for the advance of perihelion in the one-body problem. In all cases of this scalar theory, however, a zero value was obtained for the gravitational deflection of light. The only theories (scalar, vector and tensor) studied by Whitrow and Murdoch that led to the same result as General Relativity for all three Einstein tests were the tensor theories of Birkhoff (1943) and Whitehead (1922), apart from a theory due to Kustaanheimo (1957) that had so many adjustable parameters that agreement could be imposed *ad hoc*.

It is the purpose of this paper to present a simple Lorentz-invariant scalar theory of gravitation that leads to the same results as General Relativity for the three classical Einstein tests. The theory has some similarity to Whitehead's theory, but is much simpler due to its scalar nature. The object of studying this theory is that, although as far as the three Einstein tests are concerned it is indistinguishable from General Relativity, it is a much simpler theory, since it involves only one instead of ten gravitational potentials, and therefore merits attention.

2. Formulation of the theory

In the theory to be considered all effects are transmitted through space with velocity c (the velocity of light *in vacuo*). The paths of material particles satisfy the variational principle

$$\delta \int \mathrm{d}\sigma = 0 \tag{2.1}$$

where the 'gravitational metric' is given by

$$\mathrm{d}\sigma^2 = \mathrm{e}^{2\phi}\,\mathrm{d}t^2 - \frac{\mathrm{e}^{-2\phi}}{c^2}(\mathrm{d}\boldsymbol{r}\,.\,\mathrm{d}\boldsymbol{r}) \tag{2.2}$$

and the potential ϕ satisfies the Lorentz-invariant wave equation

$$\Box^2\phi = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi = 0. \tag{2.3}$$

The paths of photons are given by

$$d\sigma = 0. \tag{2.4}$$

3. Motions in a plane

The static point-source solution of (2.3) is given by $\phi = \alpha/r$, where α is constant and r is the radial coordinate. If we consider the weak field solution for large r, the gravitational metric (2.2) reduces to

$$d\sigma^2 = \left(1 + \frac{2\alpha}{r}\right) dt^2 - \frac{1}{c^2}(d\boldsymbol{r}\,.\,d\boldsymbol{r})$$

if terms in $1/r^2$ and $1/rc^2$ are neglected. Hence, on using (2.1) with this form of $d\sigma$ we obtain the classical Newtonian orbit for a gravitating test-particle if we take $\alpha = -GM/c^2$, where M is the mass of the source and G the constant of gravitation. Consequently, we take

$$\phi = -\frac{GM}{rc^2}. \tag{3.1}$$

As is usual in the field of a point-source, a free particle moves in a plane. Hence, to study its motion, we can take (2.2) in the simplified form

$$d\sigma^2 = e^{2\phi}\, dt^2 - \frac{e^{-2\phi}}{c^2}(dr^2 + r^2\, d\theta^2). \tag{3.2}$$

If we denote differentiation with respect to σ by a dot, the motion of a free particle may be described (cf Synge 1952) by

$$\delta\int \tfrac{1}{2}\left\{e^{2\phi}\dot{t}^2 - \frac{e^{-2\phi}}{c^2}(\dot{r}^2 + r^2\dot{\theta}^2)\right\} d\sigma = 0. \tag{3.3}$$

Also it follows that

$$1 = e^{2\phi}\dot{t}^2 - \frac{e^{-2\phi}}{c^2}(\dot{r}^2 + r^2\dot{\theta}^2). \tag{3.4}$$

It follows from equations (3.3) and (3.1) that

$$\dot{t} = e^k\, e^{-2\phi} \tag{3.5}$$

$$r^2\dot{\theta} = h\, e^k\, e^{2\phi} \tag{3.6}$$

where h, k are constants.

4. Advance of perihelion of orbit of a slowly moving particle in a weak field

From (3.1), (3.5) and (3.6) it follows that

$$\dot{t} = e^k \exp\left(\frac{2GM}{rc^2}\right) \tag{4.1}$$

$$r^2\dot{\theta} = e^k h \exp\left(-\frac{2GM}{rc^2}\right). \tag{4.2}$$

If the particle is slowly moving, for large r it follows from (2.2) and (4.1) that, writing $v = \mathrm{d}r/\mathrm{d}t$

$$1 = \dot{t}^2 \left\{1 - \frac{2GM}{rc^2} + \dots - \frac{v^2}{c^2}\left(1 + \frac{2GM}{rc^2} + \dots\right)\right\}$$

and hence $\dot{t}^2 \sim 1$ and so $\mathrm{e}^k \sim 1$. Writing $u = 1/r$, we obtain from (3.4), (4.1), and (4.2)

$$1 = \exp\left(-\frac{2GM}{rc^2}\right)\dot{t}^2 - \frac{1}{c^2}\exp\left(\frac{2GM}{rc^2}\right)\left\{\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + u^2\right\} r^4\dot{\theta}^2$$

which implies that

$$1 = \exp\left(\frac{2GM}{rc^2}\right) - \exp\left(-\frac{2GM}{rc^2}\right)\frac{h^2}{c^2}\left\{\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + u^2\right\}$$

and hence

$$\exp\left(\frac{2GM}{rc^2}\right) = \exp\left(\frac{4GM}{rc^2}\right) - \frac{h^2}{c^2}\left\{\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + u^2\right\}. \tag{4.3}$$

If we differentiate equation (4.3) with respect to u, we find that

$$\frac{h^2}{c^2}\left\{\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u\right\} = \frac{2GM}{c^2}\exp\left(\frac{4GM}{c^2}u\right) - \frac{GM}{c^2}\exp\left(\frac{2GM}{c^2}u\right)$$

and hence to order $1/c^2$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{GM}{h^2} + \frac{6G^2M^2}{h^2c^2}u. \tag{4.4}$$

From equation (4.4) we deduce that

$$u = \frac{GM}{h^2}\left(1 - \frac{6G^2M^2}{h^2c^2}\right)^{-1}\left[1 + \epsilon\cos\left\{\left(1 - \frac{6G^2M^2}{h^2c^2}\right)^{1/2}(\theta - \theta_0)\right\}\right]$$

where ϵ, θ_0 are constants of integration. To order $1/c^2$, this gives

$$u = \frac{GM}{h^2}\left(1 + \frac{6G^2M^2}{h^2c^2}\right)\left[1 + \epsilon\cos\left\{\left(1 - \frac{3G^2M^2}{h^2c^2}\right)(\theta - \theta_0)\right\}\right]$$

for which we have an advance of perihelion $6\pi G^2M^2/h^2c^2$ per revolution, which to order $1/c^2$ is the same value as given by General Relativity.

5. Gravitational deflection of light

Under the assumption that light moves in accordance with the condition $\mathrm{d}\sigma = 0$, it follows from (3.1) and (3.2) that

$$0 = \exp\left(-\frac{2GM}{rc^2}\right)\dot{t}^2 - \frac{1}{c^2}\exp\left(\frac{2GM}{rc^2}\right)(\dot{r}^2 + r^2\dot{\theta}^2). \tag{5.1}$$

On using (3.5), and (3.6) with the form of ϕ given by (3.1), equation (5.1) reduces to

$$0 = \exp\left(\frac{4GM}{rc^2}\right) - \left\{\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + u^2\right\}\frac{h^2}{c^2}. \tag{5.2}$$

On differentiating with respect to u and neglecting terms of higher order than the second in $1/c$, we find that

$$\frac{d^2u}{d\theta^2}+u=\frac{2GM}{h^2}. \tag{5.3}$$

The solution of (5.2) is

$$u=\frac{2GM}{h^2}\left\{1+\epsilon\cos(\theta-\theta_0)\right\}$$

where ϵ, θ_0 are constants of integration. This is the equation of a hyperbola, since for the path to be almost a straight line the eccentricity $\epsilon \gg 1$. The angle between the asymptotes is approximately $2/\epsilon$. The perihelion distance, R is given by $\theta-\theta_0=0$, so that

$$\frac{1}{R}=\frac{2GM}{h^2}(1+\epsilon)\sim\frac{2GM}{h^2}\epsilon.$$

Since $h=Rc\exp(2GM/Rc^2)$, given by taking $du/d\theta=0$ when $r=R$ in equation (5.2), it follows that the angular deflection of a light-ray from infinity that passes at a distance R from the centre of force is

$$\frac{2}{\epsilon}\sim\frac{4GMR}{h^2}=\frac{4GM}{c^2R}$$

which is the same result as obtained in General Relativity.

6. Conservation principles and the gravitational red shift

From equation (2.1) we see that

$$1=e^{2\phi}\left(\frac{dt}{d\sigma}\right)^2-\frac{e^{-2\phi}}{c^2}\frac{d\mathbf{r}}{d\sigma}\cdot\frac{d\mathbf{r}}{d\sigma}.$$

Since the velocity $\bar{c}$ of a photon in a gravitational field ϕ is given by $\bar{c}=c\,e^{2\phi}$, it follows that

$$c^2=\bar{c}^2e^{-2\phi}\left(\frac{dt}{d\sigma}\right)^2-v^2e^{-2\phi}\left(\frac{dt}{d\sigma}\right)^2 \tag{6.1}$$

where $v=|d\mathbf{r}/dt|$. If we associate with any particle a constant m_0, called its proper mass, then from (6.1) we have

$$\left(m_0e^{-\phi}\frac{dt}{d\sigma}\right)^2\bar{c}^2-\left(m_0e^{-\phi}\frac{dt}{d\sigma}\right)^2v^2=\text{constant}. \tag{6.2}$$

If we consider a region of space so small that a particle moving in a gravitational field in this region can be said to be travelling approximately along a straight line with constant velocity (assuming continuity of the particle path), the effects of gravity in this small region can be neglected and we can appeal to Special Relativity. Hence, according to an observer in an inertial frame in this region the inertial mass m of the particle will be related to its proper mass by the formula

$$m=m_0\left(1-\frac{v^2}{\bar{c}^2}\right)^{-1/2}$$

where $\bar{c}$ is the velocity of light in this region. Hence

$$m^2\bar{c}^2-m^2v^2=\text{constant} \tag{6.3}$$

in this region. We now assume (6.3) to hold generally, and consequently we must have

$$m = m_0\,e^{-\phi}\frac{dt}{d\sigma} \tag{6.4}$$

if equations (6.2) and (6.3) are to be compatible. On introducing the four-momentum defined by $\boldsymbol{P} = (m\bar{c}, m\boldsymbol{v})$, it follows that $\boldsymbol{P}.\boldsymbol{P}$ = constant.

We define measure of energy E in the usual way by

$$E = \int \frac{d\boldsymbol{p}}{dt}.\,d\boldsymbol{r}$$

where $\boldsymbol{p} = m\boldsymbol{v}$, the integral being taken along the particle path from some conveniently chosen initial level. On differentiating (6.3) with respect to t along the particle path

$$\bar{c}\frac{d(m\bar{c})}{dt} - \boldsymbol{v}\frac{d(m\boldsymbol{v})}{dt} = 0.$$

Hence the energy is given by $\int \bar{c}\, d(m\bar{c})$, taken from some chosen level. In the case of no field (i.e. $\phi \to 0$)

$$d\sigma^2 = dt^2 - \frac{1}{c^2}\,d\boldsymbol{r}.d\boldsymbol{r}$$

and the energy from some chosen initial level is given by

$$E = \int d\left(m_0\frac{dt}{d\sigma}c^2\right).$$

The initial level is chosen to be such that

$$E = m_0\frac{dt}{d\sigma}c^2$$

in order to correspond with Special Relativity. For the case of a free particle moving under no field, the equations of motion are given by

$$\delta\int d\sigma = 0$$

where

$$d\sigma^2 = dt^2 - \frac{1}{c^2}\,d\boldsymbol{r}.d\boldsymbol{r}.$$

Consequently, from equations (3.5) we deduce that $dt/d\sigma = e^k$ and hence

$$E = m_0\,e^k c^2.$$

In the case of a field with potential given by (3.1) it follows that on using (4.1) in formula (6.4), we get

$$\bar{c}\,d(\bar{c}m) = m_0c^2\,e^k\exp\left(-\frac{2GM}{rc^2}\right)d\left\{\exp\left(\frac{GM}{rc^2}\right)\right\}$$

$$= -m_0c^2\,e^k\frac{GM}{r^2c^2}\exp\left(-\frac{GM}{rc^2}\right)dr.$$

Hence we have the formula

$$E = -m_0 c^2 \mathrm{e}^k \exp\left(-\frac{GM}{rc^2}\right) + E_0$$

for the energy of the particle. From the above considerations in the case of no field, it follows that as $r \to \infty$, we must have $E \to m_0 c^2 \mathrm{e}^k$. Hence it follows that $E_0 = 2m_0 c^2 \mathrm{e}^k$. On neglecting terms in powers of GM/rc^2 higher than the first, we obtain the formula

$$E = m_0 c^2 \mathrm{e}^k \left(1 + \frac{GM}{rc^2}\right). \quad (6.6)$$

To obtain the gravitational red-shift formula in this theory, we assume that we can apply formula (6.6) to a photon of energy $h\nu$, where ν is the frequency of the photon and h Planck's constant. If ν_r denotes the standard frequency of any photon emitted at distance r from the centre of the field and ν_∞ denotes its limiting frequency at infinity then, if $GM/rc^2 \ll 1$

$$\frac{\nu_\infty}{\nu_r} = \frac{1}{1+(GM/rc^2)} \simeq 1 - \frac{GM}{rc^2}$$

Hence there will be an apparent red-shift of the frequency of the photon (as observed at infinity) given by

$$\frac{\delta\nu}{\nu} = -\frac{GM}{rc^2}$$

correspondingly

$$\frac{\delta\lambda}{\lambda} = \frac{GM}{rc^2}$$

where λ denotes wavelength. This is the same result as given by General Relativity.

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